EXAMPLE WAVE EQUATIONS

Let $y(x,t)$ denotes displacement of a vibrating string. If T is the tension in the string, ω is the weight per unit length, and g is acceleration due to gravity, then y satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{Tg}{\omega}\frac{\partial^2 y}{\partial x^2}, \quad 0<x<2,\ t>0.$$

boundary conditions all '0'

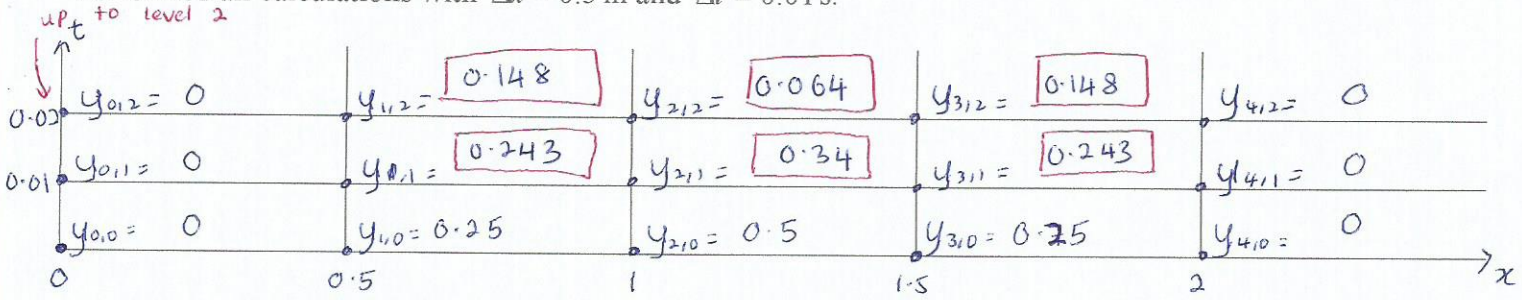
Suppose a particular string is 2 m long and is fixed at both ends. By taking $T = 1.5\text{N}$, $\omega = 0.01$ kg/m and $g = 10$ m/s^2, use the finite-difference method to solve for y up to level 2 only. The initial conditions are

$$y(x,\ 0) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{2-x}{2}, & 1 \le x \le 2 \end{cases} \quad \text{and} \quad \frac{\partial y}{\partial t}(x,\ 0) = x(x-2).$$

initial displacement (substitute into grid)

initial velocity (use for first level calculation)

Performed all calculations with $\Delta x = 0.5$ m and $\Delta t = 0.01$ s.



$$\frac{\partial^2 y}{\partial t^2} = \frac{Tg}{\omega}\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial t^2} = 1500\frac{\partial^2 y}{\partial x^2}$$

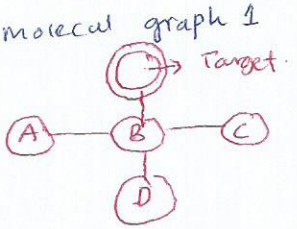
$$\frac{y_{i,j+1} - 2y_{i,j} + y_{i,j+1}}{0.01^2 \leftarrow k^2} = 1500\left(\frac{y_{i-1,j} - 2y_{i,j} + y_{i+1,j}}{0.5^2 \leftarrow h^2}\right)$$

$$y_{i,j-1} - 2y_{i,j} + y_{i,j+1} = 0.6\,(y_{i-1,j} - 2y_{i,j} + y_{i+1,j})$$

$$y_{i,j+1} = 0.6\,y_{i-1,j} + 0.8\,y_{i,j} + 0.6\,y_{i+1,j} - y_{i,j-1} \Rightarrow$$

$$= 0.6A + 0.8B + 0.6C - D$$

molecal graph 1



! But for the first level solutions we can not use molecal graph 1. Therefore, we need to substitute the given initial velocity into $y_{i,j-1}$

$$y_{i,j+1} = 0.6\,y_{i-1,j} + 0.8\,y_{i,j} + 0.6\,y_{i+1,j} - (y_{i,j-1})$$

Consider $\frac{\partial y}{\partial t} = x(x-2)$

$$\frac{y_{i,j+1} - y_{i,j-1}}{2(0.01)} = x(x-2) \quad (0.01 \leftarrow k)$$

$$y_{i,j-1} = y_{i,j+1} - 0.02\,x(x-2)$$

The new eqn.

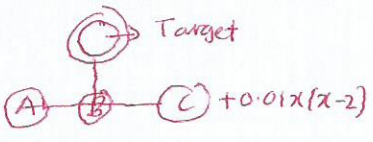
$$y_{i,j+1} = 0.6\,y_{i-1,j} + 0.8\,y_{i,j} + 0.6\,y_{i+1,j} - (y_{i,j+1} - 0.02\,x(x-2)) \Rightarrow$$

$$2y_{i,j+1} = 0.6\,y_{i-1,j} + 0.8\,y_{i,j} + 0.6\,y_{i+1,j} + 0.02\,x(x-2)$$

$$y_{i,j+1} = 0.3\,y_{i+1,j} + 0.4\,y_{i,j} + 0.3\,y_{i+1,j} + 0.01\,x(x-2)$$

$$= 0.3A + 0.4B + 0.3C + 0.01\,x(x-2)$$

molecal graph 2



* Use molecal graph 2 for first level and molecal graph 1 for other levels.